Positivity of the shape Hessian and instability of some equilibrium shapes

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Abstract: We study the positivity of the second shape derivative around an equilibrium for a 2-dimensional functional involving the perimeter of the shape and its the Dirichlet energy under volume constraint. We prove that, generally, convex equilibria lead to strictly positive second derivatives. We also exhibit some examples where strict positivity of the second order derivative holds at an equilibrium while existence of a minimum does not.

Key words: Shape Optimization, Domain Variations, Shape Hessian, Stability, Energy Functionals, Dirichlet Problem.

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1 Introduction

In this paper, we are interested in studying the positivity of the second derivative of some shape functionals at equilibrium shapes. We also exhibit some simple functionals for which strict positivity may not imply that these equilibria are local minima.

We mainly consider shape functionals involving at the same time geometric terms, like the perimeter or the measure of the shape, and also the solution of the Dirichlet problem on these shapes. We concentrate on minimization problems with prescribed measure (frequent in applications). Thus, we discuss the positivity of the second derivative of the following two-dimensional functional, defined on open subsets ω of $\mathbb{R}^2$

$$\omega \subset \mathbb{R}^2 \to E(\omega) = J(\omega) + \tau P(\omega) - \Lambda S(\omega), \tag{1.1}$$

where $\tau > 0,\ \Lambda \in \mathbb{R},\ S(\omega)$ denotes the Lebesgue measure of ω, $P(\omega)$ denotes the perimeter of ω and $J(\omega) = \int_\omega \frac{1}{2}|\nabla u_\omega|^2 - f\, u_\omega$ is the Dirichlet energy associated with the solution of the Dirichlet problem

$$u_\omega \in H_0^1(\omega),\ \ -\Delta u_\omega = f \text{ in } \omega, \tag{1.2}$$

and $f : \mathbb{R}^2 \to \mathbb{R}$ is a given function. Here, the constant Λ is to be understood as a Lagrange multiplier for the minimization of $J(\omega) + \tau\, P(\omega)$ among the sets with prescribed measure $S(\omega) = S_0$. Note that, according to (1.2)

$$J(\omega) = -\frac{1}{2}\int_\omega |\nabla u_\omega|^2 = -\frac{1}{2}\int_\omega f\, u_\omega. \tag{1.3}$$

We are interested in the positivity of the second derivative of $E(\cdot)$ around *critical shapes*, that is shapes Ω for which the solution of (1.2) satisfies also

$$\frac{1}{2}|\nabla u_\Omega|^2 - \tau\mathcal{C} + \Lambda = 0 \quad \text{on} \quad \partial\Omega, \tag{1.4}$$

where $\mathcal{C}$ denotes the curvature of $\partial\Omega$. This is the Euler-Lagrange equation of the considered minimization problem (see Section 2).

We will prove that, under natural assumptions, any *convex* equilibrium is "stable" in the sense that its second derivative is strictly positive. This remains true for shapes close to convex shapes (see Section 3). Besides this kind of results, the study of the positivity of the second derivative is interesting for itself: it involves a peculiar quadratic form on a space of functions defined on the boundary of Ω (here $H^1(\partial\Omega)$). Note that stability analysis had been made for this problem in the case $\tau = 0$ ("no surface tension") in [7], [8].

Unfortunately, in shape optimization, when positivity or "coercivity" occurs, it generally does in a weaker norm than the norm for which the functional is actually twice differentiable. In the examples above, coercivity occurs in H^1 while second order Taylor formulas are valid only in spaces like C^2 or even $C^{2,\alpha}$. In general, this is not sufficient to guarantee that there is a local minimum, even in the stronger norm. However, for shape optimization of the type considered here, the situation is slightly better and, as proved in [5],[4], positivity does imply existence of a strict local minimum with respect to the stronger norms of C^2-type, whence the interest of studying positivity. It is likely that this minimum is local for a weaker norm, but it is not yet understood how much "weaker" this norm may be.

As a contribution to the latter question, we give next "bad" examples where positivity of the second derivative occurs while "small" perturbations may strictly decrease the energy. This is done for simple functionals of the form $F(\omega) = \int_\omega g(u_\omega)$ and around circles. Perturbations, although "small", are rather irregular since they consist in modifying the circle into "close" annuli. In particular, they cannot be represented as normal perturbations to the circle. This says that the second classical shape derivative does not provide all information about existence of minima and should probably include extra information coming from the so-called topological derivative (see e.g. [16]).

Section 2 is devoted to the computation of the second derivatives of the functionals of type (1.1). This is not a quite easy step in general. We mainly refer to [17] where the structure of any second shape derivative is given. This a priori knowledge helps quite a lot for the computations. Section 3 concerns the study of the positivity of the second derivative of the functionals (1.1). Section 4 concerns the particular case where the equilibrium shape is a disk (in which case a more complete analysis may be made). Section 5 describes two examples where strict positivity does not imply existence of a local minimum.

2 Computation of the second derivative of $E(\cdot)$

We consider the functional $E(\cdot)$ given by (1.1-1.2) and we assume that f is regular (to make it simple, we will assume that it is at least bounded and in $C^1(\mathbb{R}^2)$). As proved in several places (see e.g. [8],[7],[19],[15]), this functional is twice differentiable *around regular enough shapes* Ω. Sharp differentiability may be obtained in Hölder spaces of type $C^{2,\alpha}, \alpha \in (0,1)$. Since, we are mainly interested here in structures of derivatives, we will make it simple and work in C^3-spaces. We denote by Θ_3 the space of functions $\theta \in C^3(\mathbb{R}^2, \mathbb{R}^2)$ whose derivatives up the the third order are bounded. This space is equipped with its natural norm. Then the mapping

$$\mathcal{E} : \theta \in \Theta_3 \to E\big((I+\theta)(\Omega)\big), \tag{2.1}$$

is twice differentiable at $\theta = 0$ (here I denotes the identity in Θ_3).

We will compute these derivatives around a *regular* shape Ω that we will assume to be a bounded open subset of $\mathbb{R}^2$ with regular boundary $\Gamma = \partial\Omega$, say of class C^5 at least. For this, let us first recall the structure of the first and second (Fréchet) shape derivatives in general. It is well-known for first derivatives from the pioneer paper by Hadamard [13]. It may also be found for the second derivative under different forms in several papers (see [15],[18],[6],[2],[19],[17]). Here, we use the notations and results of [17], Theorem 2.1 and Corollaries 2.4,2.9. We state it for our functional $E(\cdot)$, but it is the same for any regular functional. We denote by ν the normal unit exterior derivative to $\Gamma = \partial\Omega$. The Fréchet-derivatives of $\mathcal{E}(\cdot)$ at $\theta = 0$ will be described by their action on arbitrary regular displacements $\xi, \eta \in \Theta_3$.

Lemma 2.1 *There exists a continuous linear map l_1 from $C^3(\Gamma)$ into $\mathbb{R}$ such that, for all $\xi \in \Theta_3$,*

$$\mathcal{E}'(0)(\xi) = l_1(\xi \cdot \nu).$$

There exists a continuous bilinear symmetric map l_2 from $C^3(\Gamma) \times C^3(\Gamma)$ into $\mathbb{R}$ such that for all $\xi, \eta \in \Theta_3$,

$$\mathcal{E}''(0)(\xi,\eta) = l_2(\xi\cdot\nu, \eta\cdot\nu) - l_1(\xi^\tau \cdot D^\tau \nu\, \eta^\tau + \nu \cdot D^\tau \xi\, \eta^\tau + \nu \cdot D^\tau \eta\, \xi^\tau),$$

where $\xi^\tau = \xi - (\xi\cdot\nu)\nu$ denotes the tangential component of ξ and where $D^\tau \xi = D(\xi^\tau)$ denotes the matrix of the derivatives of ξ^τ.

If Ω is a critical shape, that is $l_1 = 0$, then the second derivative depends only on l_2 and on the normal components of the displacements of ξ, η at the boundary Γ. It is also the case for any Ω if ξ, η are normal to Γ (since $\xi^\tau = \eta^\tau = 0$). As a consequence of this structure theorem, to obtain full information on derivatives, it is sufficient to identify l_1, l_2. This may be done by using specific variations of the form $t \to E(\Omega_t)$ where t tends to 0 and where the Ω_t's are well-chosen perturbations of Ω (chosen to allow as simple computations as possible).

One may for instance choose $\Omega_t = T_t(\Omega)$ where $T_t : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as follows and based on displacements which are *normal* to Γ: For any regular function $h : \Gamma \to \mathbb{R}$, we denote by $\widetilde{h}$ a regular extension of h to $\mathbb{R}^2$ and by $\widetilde{\nu}$ a regular unitary extension of ν to $\mathbb{R}^2$(see e.g. [12] for details). Now set

$$\left.\begin{array}{rl} \forall x \in \mathbb{R}^2, \quad \forall t \geq 0: & T_t x = x + t\,\widetilde{h}(x)\widetilde{\nu}(x), \\ \Omega_t = T_t(\Omega), & e(t) = E(T_t(\Omega)) = E(\Omega_t). \end{array}\right\} \tag{2.2}$$

Then, since $e(t) = \mathcal{E}(t\,\widetilde{h\nu})$, by the chain rule and Lemma 2.1, we obtain

$$e'(0) = l_1(h), \;\; e''(0) = l_2(h,h). \tag{2.3}$$

Now, we can describe explicitly $e'(0), e''(0)$ in our case. We use the following notations:

$$u = u_\Omega, \;\; \beta = \nabla u \cdot \nu = \partial u / \partial \nu.$$

We denote by h' the derivative with respect to the length parameter on Γ and by D is the pseudo-differential operator on Γ which, to each regular function $h : \Gamma \to \mathbb{R}$, associates the trace on Γ of $\partial H/\partial \nu$ where H is the harmonic extension of h to Ω. In particular, we have

$$\int_\Gamma hDh = \int_\Omega |\nabla H|^2. \tag{2.4}$$

It easily follows from this identity that D extends to a continuous operator from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$. Recall also that $\mathcal{C}$ denotes the curvature of Γ seen from inside Ω.

Theorem 2.2

$$e'(0) = l_1(h) = \int_\Gamma (-\frac{1}{2}\beta^2 + \tau\mathcal{C} - \Lambda)h, \tag{2.5}$$

$$e''(0) = l_2(h,h) = \int_\Gamma \beta h D(\beta h) + (\mathcal{C}\beta^2/2 - \Lambda\mathcal{C} + f\,\beta)h^2 + \tau h'^2. \tag{2.6}$$

Remark 2.1 If Ω is a critical shape for $\omega \to J(\omega) + \tau P(\omega)$ (see 1.1) under the constraint $S(\omega) = S_0$, then there exists a real number Λ such that $l_1 \equiv 0$, that is

$$-\frac{1}{2}\beta^2 + \tau\mathcal{C} - \Lambda = 0 \quad \text{on} \quad \Gamma, \tag{2.7}$$

which is the relation (1.4) announced in the introduction (note that, since $u = 0$ on Γ, then $|\nabla u| = |\partial u/\partial \nu| = |\beta|$). In (2.6), we may replace β by its expression given in (2.7). As a consequence, around an equilibrium, the expression of $e''(0)$ depends only on the geometry of Ω, on Λ and on the data τ, f.

The result of Theorem 2.2 will be obtained by differentiating each of the three terms in the total energy E. Most of the needed computations are available in the literature under various forms. We will mainly refer to [17] where the three terms of our functional $E(\cdot)$ are considered and we will indicate the necessary additional information.

Proof of Theorem 2.2: According to Lemma 2.1, let us denote by $l_i^S, l_i^P, l_i^M, i = 1, 2$ the linear and bilinear forms respectively describing the first and second derivatives of each of the functionals

$$\Omega \to S(\Omega),\ P(\Omega),\ \int_\Omega |\nabla u_\Omega|^2.$$

Note that, by (1.1), (1.3), for $i = 1, 2$

$$l_i = -\frac{1}{2} l_i^M + \tau l_i^P - \Lambda l_i^S. \tag{2.8}$$

By [17], Proposition 5.1, we have

$$l_1^S(h) = \int_\Gamma h,\ \ l_2^S(h,h) = \int_\Gamma \mathcal{C} h^2, \tag{2.9}$$

$$l_1^P(h) = \int_\Gamma \mathcal{C} h,\ \ l_2^P(h,h) = \int_\Gamma |\nabla^\tau h|^2$$

where ∇^τ denotes the tangential gradient,

$$l_1^M(h) = \int_\Gamma |\nabla u|^2 h,\ \ l_2^M(h,h) = \int_\Gamma -2\omega(h)\frac{\partial \omega(h)}{\partial \nu} + h^2[2\frac{\partial u}{\partial \nu}\frac{\partial^2 u}{\partial \nu^2} + \mathcal{C}(\frac{\partial u}{\partial \nu})^2],$$

where $\omega(h)$ is the solution of

$$\Delta \omega(h) = 0 \text{ in } \Omega,\ \ \omega(h) = -h\frac{\partial u}{\partial \nu} \text{ on } \Gamma. \tag{2.10}$$

Formula (2.5) follows from these expressions together with (2.8), as well as formula (2.6) with the extra help of the following technical lemma:

Lemma 2.3

$$|\nabla^\tau h|^2 = {h'}^2, \tag{2.11}$$

$$\omega(h)\frac{\partial \omega(h)}{\partial \nu} = \beta h D(\beta h), \tag{2.12}$$

$$\frac{\partial u}{\partial \nu}\frac{\partial^2 u}{\partial \nu^2} = -\mathcal{C}\beta^2 - f\beta. \tag{2.13}$$

Proof: To obtain (2.11), recall that the tangential gradient is given by

$$\nabla^\tau h = \nabla\widetilde{h} - (\nabla\widetilde{h} \cdot \widetilde{\nu})\widetilde{\nu}.$$

Let $s \to X(s)$ be a parametrization of Γ in terms of the length parameter s. Then, $X'(s)$ is a unit vector tangent to Γ at $X(s)$ so that

$$\nabla^\tau h \cdot X'(s) = \nabla\widetilde{h} \cdot X'(s) = \frac{d}{d\,s}\tilde{h}(X(s)) = \frac{d}{d\,s}h(X(s)),$$

and $|\nabla^\tau h|^2 = |\nabla^\tau h \cdot X'(s)|^2 = h'(s)^2$ if we simply set $h(s) = h(X(s))$.

For (2.12), we introduce the solution u_t of the Dirichlet problem:

$$-\Delta u_t = f \text{ in } \Omega_t, \;\; u_t = 0 \text{ on } \partial\Omega_t \;\text{ or }\; u_t(T_t(x)) = 0 \;\text{ on }\; \Gamma. \tag{2.14}$$

By differentiating this system with respect to t at $t = 0$, we obtain

$$\Delta(\partial_{t_{|t=0}} u_t) = 0 \text{ in } \Omega, \;\; \partial_{t_{|t=0}} u_t = -h\frac{\partial u}{\partial \nu} \text{ on } \Gamma. \tag{2.15}$$

It follows that:

$$\partial_{t_{|t=0}} u_t = \omega(h) \text{ and } \partial\omega(h)/\partial\nu = D(-\beta h), \quad \text{whence (2.12).}$$

To obtain (2.13), we may argue as in [7],[8]) using local coordinates. Assume that $(0,0) \in \Gamma$ and Γ is defined as the graph of a regular fonction $\theta :]-\eta, \eta[\to \mathbb{R}$ satisfying $\theta(0) = \theta'(0) = 0$. Moreover, Ω is locally *below* the graph of θ (so that $\partial/\partial\nu = \partial/\partial y$). Then, differentiating twice $u(x, \theta(x)) = 0$ with respect to x, we obtain:

$$\begin{aligned} &u_x(x,\theta(x)) + u_y(x,\theta(x))\theta'(x) = 0, \\ &u_{xx}(x,\theta(x)) + 2\, u_{xy}(x,\theta(x))\theta'(x) + u_{yy}(x,\theta(x)){\theta'}^2(x) + u_y(x,\theta(x))\theta''(x) = 0. \end{aligned}$$

We deduce that at $x = 0$ (recall that u is at least C^2)

$$u_{xx}(0,0) + u_y(0,0)\theta''(0) = 0$$

which, together with $-\Delta u = f$ *on* Ω implies that, on the boundary

$$\frac{\partial u}{\partial \nu}\frac{\partial^2 u}{\partial \nu^2} = u_y\, u_{yy} = u_y(-u_{xx} - f) = (u_y)^2\theta''(0) - f\, u_y.$$

Since $\mathcal{C}$ is the curvature of Γ seen from below, we have $\mathcal{C} = -\theta''(0)$ at the origin, whence (2.13).

3 Positivity of second derivative

We use here the notations of Section 2. Let $(\Omega, u = u_\Omega)$ be a regular equilibrium (or a critical shape) for the minimization of

$$\omega \to J(\omega) + \tau P(\omega) = -\frac{1}{2}\int_\omega |\nabla u_\omega|^2 + \tau P(\omega)$$

under the constraint $S(\omega) = S_0$ where S_0 is given. This means that there exists a real number Λ (a Lagrange multiplier) such that the derivative of $E(\cdot) = J(\cdot) + \tau P(\cdot) - \Lambda S(\cdot)$ vanishes at Ω. With the previous notations and definitions (1.1,2.1), we have $\mathcal{E}'(0) = 0$: according to Section 2, this means that

(2.7) holds on $\Gamma = \partial\Omega$, that is $\beta^2 = 2(\tau\mathcal{C} - \Lambda)$.

A necessary second order condition: Assume now that $J(\cdot) + \tau P(\cdot)$ has a local minimum at Ω among the regular domains ω satisfying the measure constraint $S(\omega) = S_0$. Then $\theta \in \Theta_3 \to (J + \tau P)((I + \theta)(\Omega))$ has a minimum at $\theta = 0$ on the manifold of θ's such that $S\big((I+\theta)(\Omega)\big) = S(\Omega)$. As a consequence, the second derivative of the full Lagrangian $\theta \to (J + \tau P - \Lambda S)\big((I + \theta)(\Omega)\big)$ is nonnegative on the hyperplane of linearized constraints, that is on $\{\xi \in \Theta_3;\ \int_\Gamma \xi \cdot \nu = 0\}$. This means that, for all $h \in C^3(\Gamma)$ such that $\int_\Gamma h = 0$, the quadratic form

$$Q(h) = \int_\Gamma \beta h D(\beta h) + (\tau\mathcal{C} - 2\Lambda)\mathcal{C}h^2 + f\,\beta h^2 + \tau {h'}^2 \tag{3.1}$$

is nonnegative.

Remark 3.1 Since Ω is bounded and regular, we may assume that its boundary is the union of disjoint regular Jordan curves $\Gamma_i, i = 1, ...p$ (*a Jordan curve is, by definition, the one-to-one continuous image of the unit circle*; in particular, it is a closed curve). Choosing in (3.1) test-functions h vanishing on all Γ_i but one, we obtain that, for all $i = 1, ...p$ and for all $h \in C^3(\Gamma_i)$ with $\int_{\Gamma_i} h = 0$, we have

$$Q_i(h) = \int_{\Gamma_i} \beta h D(\beta h) + (\tau\mathcal{C} - 2\Lambda)\mathcal{C}h^2 + f\,\beta h^2 + \tau {h'}^2 \geq 0. \tag{3.2}$$

Remark 3.2 In the expression of Q_i, we must remember that the operator D depends on the set Ω and not only on Γ_i. Note that, by (2.4) and classical Poincaré inequality, there exists $\delta > 0$, depending only on Ω, such that

$$\forall\, h \in Z_0^1(\Gamma),\ \int_\Gamma h\, Dh = \int_\Gamma |\nabla H|^2 \geq \delta \|h\|^2_{L^2(\Gamma)}. \tag{3.3}$$

Therefore, the contribution of the term $h\,Dh$ helps making Q positive in any case.

The coercivity question: We will now concentrate on the study of the positivity of the quadratic form $h \to Q(h)$ for functions h satisfying $\int_\Gamma h = 0$ when Γ *is a regular Jordan curve.* Then Ω is the simply connected interior of Γ.

Obviously, $Q(\cdot)$ is continuous on the functional space $H^1(\Gamma)$ and the H^1-norm clearly appears as the "optimal" one for this continuity and also for the question of positivity or "coercivity". Therefore, the natural question is to decide whether or not $Q(\cdot)$ is *coercive* on the space

$$Z_0^1(\Gamma) = \{u \in H^1(\Gamma) : \int_\Gamma h = 0\}$$

that is

$$\exists \eta > 0; \forall h \in Z_0^1(\Gamma),\ Q(h) \geq \eta \|h\|^2_{H^1(\Gamma)}. \tag{3.4}$$

As explained in the introduction, if H^1-coercivity holds, then the functional $J(\cdot)$ has a strict local minimum at Ω at least in a $C^{2,\alpha}$-topology, i.e. among $C^{2,\alpha}$-perturbations of Ω preserving the volume.

We will mainly concentrate on the case where

$$\left.\begin{array}{c} S\big(\{x \in \mathbb{R}^2;\ u_\Omega(x) \neq 0\ a.e.\ \}\big) = S(\Omega) = S_0, \\ f \text{ is compactly supported in } \Omega, \\ \Gamma \text{ is a regular Jordan curve with interior } \Omega. \end{array}\right\} \tag{3.5}$$

The first condition is just a way to say that the constraint $S(\Omega) = S_0$ is "saturated", otherwise we would work with the smaller open subset $\Omega' = \{x \in \mathbb{R}^2; u_\Omega(x) \neq 0\}$. The second condition is natural in applications (see [14],[7],[8],[3]). More comments will be made later on the case where f does not vanish around Γ (see Remark 3.7).

Two families of situations: It turns out that there will be two rather different families of situations: indeed, the equilibrium condition (2.7) writes

$$\tau\mathcal{C} - \Lambda = \frac{1}{2}|\nabla u|^2 \geq 0 \quad \text{on} \quad \Gamma, \tag{3.6}$$

so that, if $\mathcal{C}_m := \min_\Gamma \mathcal{C}$, then

$$\tau\mathcal{C}_m \geq \Lambda.$$

But the situation will be different depending on whether this inequality is strict or not. If there exists a point of Γ at which $\nabla u = 0$, then necessarily $\tau\,\mathcal{C}_m = \Lambda$. This is necessarily the case if, for instance, $\int_\Omega f = 0$ (Γ being a Jordan curve). Indeed, if ∇u did not vanish on Γ, then, we would have

$$\int_\Gamma |\nabla u| = |\int_\Gamma \nabla u \cdot \nu| = |\int_\Omega \Delta u| = |\int_\Omega f| = 0,$$

which would be a contradiction with $\nabla u \neq 0$ on Γ. In some applications, like the shaping of liquid metals by electromagnetic devices, this case $\tau\mathcal{C}_m = \Lambda$ turns out to be more "physical" since, in two-dimensional models, it is natural to assume that the distribution of currents f satisfies $\int_\Omega f = 0$ (see [7],[8],[14]).

A first case: $\tau\,\mathcal{C}_m = \Lambda$ (and (3.5)). Then, the form Q writes

$$Q(h) = \tau \int_\Gamma \gamma h D(\gamma h) + (\mathcal{C} - 2\mathcal{C}_m)\mathcal{C}h^2 + h'^2, \tag{3.7}$$

where $\gamma^2 = 2(\mathcal{C} - \mathcal{C}_m)$.

A main remark is that, in this situation, the positivity of Q, and consequently the stability of Ω, depend *only on the geometry of the boundary* Γ.

Positivity may be obtained in the following case. We denote by $L(\Gamma)$ the length of Γ. We recall that $\mathcal{C}$ denotes the curvature of the Jordan curve Γ *seen from inside* Ω.

Theorem 3.1 *Assume $\tau\mathcal{C}_m = \Lambda$. Then, if Γ is a convex Jordan curve (i.e. $\mathcal{C}_m \geq 0$), or, more generally, if Γ is a Jordan curve satisfying*

$$\mathcal{C}_m > -2\pi/L(\Gamma), \tag{3.8}$$

then, H^1-coercivity condition (3.4) holds.

Remark 3.3 The case where Γ is a circle will be considered in Section 4. Note that, under the above assumptions, Γ cannot be a circle. Indeed,

$$\beta^2 = 2(\tau\mathcal{C} - \Lambda) = 0,\ \mathcal{C} \equiv \mathcal{C}_m,\ \tau\mathcal{C}_m - \Lambda = 0,$$

imply that $0 \equiv \beta = \partial u/\partial\nu$ *on* Γ. But the three conditions "$u = 0, \nabla u = 0$ on Γ and $\Delta u = 0$ in a neighborhood of Γ" (see (3.5)) would imply that u be identically equal to 0 in a neighborhood of Γ which is not the case by (3.5).

Remark 3.4 Recall that, since Γ is a Jordan curve and since $\mathcal{C}$ is its curvature *seen from inside* Γ, then $\int_\Gamma \mathcal{C} = 2\pi$. This implies

$$2\pi = \int_\Gamma \mathcal{C} \geq L(\Gamma)\mathcal{C}_m. \tag{3.9}$$

Here, this inequality is strict since Γ is not a circle. Therefore, under the assumptions of above Theorem, we even have

$$\mathcal{C}_m^2 < 4\pi^2/L^2(\Gamma). \tag{3.10}$$

To prove Theorem 3.1, we use the following lemma:

Lemma 3.2

$$\forall h \in Z_0^1(\Gamma), \qquad \int_\Gamma h'^2 \geq \frac{4\pi^2}{L^2(\Gamma)} \int_\Gamma h^2. \tag{3.11}$$

Proof: Let

$$\lambda = \inf\{\int_\Gamma h'^2; h \in H^1(\Gamma), \int_\Gamma h = 0, \int_\Gamma h^2 = 1\}.$$

By compactness of the embedding of $H^1(\Gamma)$ into $L^2(\Gamma)$, a minimizing subsequence in above definition converges to $u \in Z_0^1(\Gamma)$ realizing the infimum. Differentiating

$$t \to \int_\Gamma \frac{(u' + th')^2}{\int_\Gamma (u + th)^2},$$

at $t = 0$ leads, in a classical way, to the fact that λ is the smallest positive eigenvalue of $u \to -u''$ on $[0, L(\Gamma)]$ with periodic conditions. We easily check that corresponding eigenfunctions are $\cos 2\pi s/L(\Gamma), \sin 2\pi s/L(\Gamma)$, so that $\lambda = 4\pi^2/L^2(\Gamma)$. Assertion (3.11) follows.

Proof of Theorem 3.1: By (3.7,3.3), we have

$$Q(h) \geq \tau \int_\Gamma [2\delta(\mathcal{C} - \mathcal{C}_m) + (\mathcal{C} - 2\mathcal{C}_m)\mathcal{C}]\, h^2 + h'^2.$$

Since the function $x \to 2\delta(x - \mathcal{C}_m) + (x - 2\mathcal{C}_m)x$ is increasing on $[\mathcal{C}_m, +\infty[$, this implies

$$Q(h) \geq \tau \int_\Gamma h'^2 - \mathcal{C}_m^2 h^2. \tag{3.12}$$

According to (3.10), let $\epsilon > 0$ such that $(1-\epsilon)(2\pi/L)^2 = \mathcal{C}_m^2$. Then, by (3.12) and Lemma 3.2

$$Q(h) \geq \tau \int_\Gamma \epsilon h'^2 = \tau\epsilon \|h\|^2_{H^1(\Gamma)}.$$

Second case: $\tau\mathcal{C}_m > \Lambda$ (and (3.5)).

Theorem 3.3 *Assume $\tau\mathcal{C}_m > \Lambda$. If Γ is a convex Jordan curve, or more generally if*

$$\mathcal{C}_m > -\min\{\delta, 2\,\pi/L(\Gamma)\}, \tag{3.13}$$

then H^1-coercivity condition (3.4) holds.

Proof of Theorem 3.3 Set $\hat{\delta} = \min\{\delta, 2\,\pi/L(\Gamma)\}$ (recall that δ is defined in (3.3)). Then, using (3.1), (3.3) and (3.6), we have

$$Q(h) \geq \int_\Gamma [2\,\hat{\delta}(\tau\mathcal{C} - \Lambda) + (\tau\mathcal{C} - 2\Lambda)\mathcal{C}]h^2 + \tau h'^2. \tag{3.14}$$

The function $x \to \psi(x) = 2\hat{\delta}(\tau x - \Lambda) + (\tau x - 2\Lambda)x + \tau\hat{\delta}^2$ is stricly increasing on $[\Lambda/\tau, +\infty[$. Note that $\psi(-\hat{\delta}) = 0$. Let $\epsilon > 0$ be such that $\mathcal{C}_m \geq \epsilon - \hat{\delta}$ and let $\hat{\epsilon}$ be so that $\psi(\epsilon - \hat{\delta}) = \hat{\epsilon}\tau\hat{\delta}^2 (> 0)$. By Lemma 3.2 and the various definitions $\hat{\delta}, \epsilon, \hat{\epsilon}$, we have

$$Q(h) \geq \int_\Gamma [2\hat{\delta}(\tau\mathcal{C} - \Lambda) + (\tau\mathcal{C} - 2\Lambda)\mathcal{C} + \tau(1-\hat{\epsilon})\hat{\delta}^2]h^2 + \hat{\epsilon}\tau h'^2 \geq \int_\Gamma \hat{\epsilon}\tau h'^2.$$

Remark 3.5 The assumption (3.13) in Theorem 3.3 is probably not optimal to obtain positivity. However, if Γ is too far from a convex curve, then positivity may fail.

Proposition 3.4 *There exist non convex analytic Jordan curves Γ for which the associated quadratic form Q has directions of instability when τ is small enough.*

Proof: Assume that Γ is not convex so that C_m and Λ are strictly negative ($0 > \tau\mathcal{C}_m > \Lambda$). The function $\beta = \{2(\tau\mathcal{C} - \Lambda)\}^{1/2}$ converges in C^1 to the constant function $(2|\Lambda|)^{1/2}$ as τ tends to 0. As a consequence, for $h \in H^{1/2}(\Gamma)$, βh tends to $(2|\Lambda|)^{1/2}h$ in $H^{1/2}(\Gamma)$ and $D(\beta h)$ tends to $(2|\Lambda|)^{1/2}Dh$ in $H^{-1/2}(\Gamma)$. Then, $Q(h)$ tends to $Q_0(h)$ given by

$$Q_0(h) = 2|\Lambda| \int_\Gamma hDh + \mathcal{C}h^2.$$

To conclude, it is sufficient to prove that we may choose Γ and $h \in H^{1/2}(\Gamma)$ with $\int_\Gamma h = 0$ and such that $Q_0(h) < 0$.

We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Let Φ be a conformal mapping from the unit disk Ω_0 onto Ω which sends the unit circle Γ_0 on Γ in a one-to-one way. Since Γ is analytic, Φ extends to a very regular function on Γ. If we denote by D_0 the operator D when Γ is replaced by Γ_0, and if $\tilde{h} := h \circ \Phi$, then

$$D_0\tilde{h} = [Dh \circ \Phi]|\Phi'|,$$

so that

$$Q_0(h) = 2|\Lambda| \int_{\Gamma_0} \tilde{h} D_0 \tilde{h} + [\mathcal{C} \circ \Phi]|\Phi'|h^2.$$

It is sufficient to prove that we may choose the function $[\mathcal{C} \circ \Phi]|\Phi'|$ (i.e. Γ) in such a way that the above expression for Q_0 becomes negative for some h.

Choose for instance $\Phi'(z) = e^{-az^2}$. It can be checked that, for $0 < a < 2.478$, Φ is one-to-one from the unit disk into its image (since Φ is locally injective and the image of the unit circle has no double point). For $\theta \in [0, 2\pi]$, $\mathcal{C}\big(\Phi(e^{i\theta})\big)|\Phi'(e^{i\theta}|$ is given as the real part of

$$1 + e^{i\theta}\Phi''(e^{i\theta})/\Phi'(e^{i\theta}) = 1 - 2a\cos 2\theta.$$

For this choice, since $D_0(cos\,\theta) = cos\,\theta$, we find

$$Q_0(cos(\cdot)) = 2|\Lambda| \int_0^{2\pi} cos^2\,\theta[2 - 2cos\,\theta] = 2|\Lambda|\pi(2 - a).$$

Therefore, any value $a \in (2, 2.478)$ gives the desired example.

Remark 3.6 From the analysis in [14], we prove that, for a given analytic closed Jordan curve, and for τ, Λ given such that $0 < \tau, \Lambda < \tau\mathcal{C}_m$, we may find a function f *compactly supported inside* Γ such that Γ be an equilibrium shape for the corresponding functional $E(\cdot)$. Therefore, it makes sense to analyze the stability of the given analytic Jordan curve Γ in terms of the parameters τ, Λ.

Remark 3.7 About the influence of f: From the previous remark, we see that the case of f being compactly supported in Ω is actually very general. Now that we have analyzed the form Q without the term in f, we might want to understand the influence of adding it: it writes $\int_\Gamma f\beta h^2$ where $\beta = \partial u/\partial\nu$. If, for instance, $f \geq 0$, then $u \geq 0$ so that $\beta \leq 0$ and $\int_\Gamma f\beta h^2 \leq 0$. This suggests that the presence of f makes Q "less positive". We could state results involving the respective values of $\mathcal{C}_m$ and f, but, in any case, a convex curve will not be stable for all f's. A very detailed analysis of this is made in the next Section where Γ is a disk, in which case precise explicit computations can be made.

Remark 3.8 If $\partial\Omega$ is the disjoint union of convex Jordan curves (which indeed happens in applications), then stability condition (3.4) holds.

4 The particular case of the disk

We assume in all this section that *the equilibrium shape* Ω *is the unit disk.* We are able to precisely study the positivity of the second derivative through the use of Fourier series expansions. Note that, if the solution u_Ω is radial ($u = u(r)$), then the disk is indeed an equilibrium shape since we then have on the boundary Γ:

$$|\nabla u|^2 = u'(1)^2 = constant,$$

so that by setting $\Lambda = \tau - u'(1)^2/2$, we do have the equilibrium condition (2.7). Here $\beta = u'(1), \beta^2 = 2(\tau - \Lambda)$.

Note that u is radial as soon as f is itself radial: this will be assumed in this section: $f = f(r)$.

The second derivative writes (see (3.1))

$$Q(h) = \int_\Gamma \beta^2 h D(h) + [\tau - 2\Lambda + \beta f(1)]h^2 + \tau {h'}^2. \tag{4.1}$$

For h given in $Z_0^1(\Gamma)$, we introduce its Fourier's series expansion

$$h = h(\theta) = \sum_{n\geq 1} (a_n \cos n\theta + b_n \sin n\theta) \tag{4.2}$$

(Here $a_0 = 0$ since $\int_\Gamma h = 0$).

Lemma 4.1

$$Q(h) = \pi \sum_{n\geq 1} [\tau n^2 + \beta^2 n + (\tau - 2\Lambda + \beta f(1))](a_n^2 + b_n^2), \tag{4.3}$$

where $\beta = -\int_0^1 r f(r) dr, \beta^2 = 2(\tau - \Lambda)$.

Proof: By direct integration, we obtain $r\, u'(r) = -\int_0^r s f(s)\, ds$, whence the expression of β.

To compute Dh, we first note that the harmonic extension of h on the disk is given by $\sum_{n\geq 1} r^n (a_n \cos n\theta + b_n \sin n\theta)$ so that

$$Dh(\theta) = \sum_{n\geq 1} n(a_n \cos n\theta + b_n \sin n\theta).$$

Easy computations lead to

$$\int_\Gamma h D(h) = \pi \sum_{n\geq 1} n(a_n^2 + b_n^2), \int_\Gamma h^2 = \pi \sum_{n\geq 1} (a_n^2 + b_n^2), \int_\Gamma {h'}^2 = \pi \sum_{n\geq 1} n^2(a_n^2 + b_n^2).$$

We combine these equalities according to (4.1) to obtain (4.3).

To state the positivity result, we introduce the expression

$$\sigma = 2\Big[\int_0^1 r f(r)\, dr\Big]^2 - f(1) \int_0^1 r f(r)\, dr.$$

Theorem 4.2 *If $\sigma > 0$, H^1-coercivity (3.4) holds and more precisely*

$$Q(h) \geq \pi \min\{\sigma, \tau\} \sum_{n\geq 1} n^2(a_n^2 + b_n^2). \tag{4.4}$$

If $\sigma < 0$, the disk is unstable.
If $\sigma = 0$, then Q in nonnegative on $Z_0^1(\Gamma)$; moreover, there exists $\delta > 0$ such that, for all $h \in Z_0^1(\Gamma)$ with $a_1 = b_1 = 0$, we have

$$Q(h) \geq \delta\pi \sum_{n\geq 2} n^2(a_n^2 + b_n^2). \tag{4.5}$$

Remark 4.1 When f is constant, since the functional $E(\cdot)$ is invariant by translations, one cannot expect coercivity in the directions of translations, i.e, on the subspace of $Z_0^1(\Gamma)$ spanned by $\{cos\theta, sin\theta\}$. We check that, in this case, $\sigma = 0$. However, above theorem says that H^1-stability nevertheless holds for deformations leaving invariant the center of the disk ($a_1 = 0, b_1 = 0$).

Remark 4.2 If f is compactly supported in the disk, then $f(1) = 0$ and $\sigma \geq 0$; since it cannot be 0 (see Remark 3.3), the disk is always stable, no matter the radial distribution of "charges" (or "current") inside the disk. On the other hand, if $f(1)$ is large compared to $\int_0^1 rf(r)\,dr$, then instabilities appear. If we refer to the underlying application arising in the electromagnetic shaping of liquid metals, having $f(1) \neq 0$ means that we have inductors touching the liquid metal at its boundary. We do believe that this may create instabilities.

Remark 4.3 Note that σ depends only on f. Therefore, if $\sigma > 0$, stability holds no matter the value of $\tau > 0$. If $\tau = 0$, then coercivity still occurs, but only in the norm $H^{1/2}(\Gamma)$ (see the expression of $\mathcal{R}(n)$ in the following proof). We refer to [8],[7],[5] for results in this case.

Proof of Theorem 4.2: Note that $\tau - 2\Lambda = -\tau + \beta^2$. The sequence

$$n \to \mathcal{R}(n) := \tau n^2 + \beta^2 n + (-\tau + \beta^2 + \beta f(1))$$

is increasing, so that $\mathcal{R}(n) > 0$ for any $n \geq 1$ as soon as $\mathcal{R}(1) > 0$, that is when

$$\sigma = 2\beta^2 + \beta f(1) > 0.$$

More precisely, since the mapping

$$n \to (\tau - \min\{\tau, \sigma\})n^2 + \beta^2 n,$$

is increasing, we even have

$$\forall\, n \geq 1,\ \ \mathcal{R}(n) \geq \min\{\sigma, \tau\} n^2,$$

which proves coercivity of Q for the H^1-norm in $Z_0^1(\Gamma)$.

On the other hand, if $\sigma < 0$, $\mathcal{R}(1) < 0$, so that instability occurs at least in the direction of the subspace spanned by $\{cos\theta, sin\theta\}$.

Now, if $\sigma = 0$, $\mathcal{R}(1) = 0$ and $\forall n \geq 2, \mathcal{R}(n) > 0$ so that Q is nonnegative. More precisely, if $\delta = \min\{(3\tau + \beta^2)/4, \tau\}$, then

$$\forall n \geq 2,\ \mathcal{R}(n) = \tau n^2 + \beta^2 n - \tau - \beta^2 \geq \delta n^2.$$

This proves the coercivity of Q for the H^1-norm on the subspace of $h \in Z_0^1(\Gamma)$ such that $a_1 = b_1 = 0$.

5 Some counterexamples

Let $g : \mathbb{R} \to \mathbb{R}$ be a function of class C^2. We consider the shape functional $\omega \to \int_\omega g(u_\omega)$ under the constraint $S(\omega) = S_0$, where $u = u_\omega$ is simply the so-called stress function, solution of

$$-\Delta u = 1 \text{ in } \omega,\ u = 0 \text{ on } \partial\omega.$$

We introduce the corresponding Lagrangian

$$L(\omega) = \int_\omega g(u_\omega) - \Lambda S(\omega)$$

where Λ is the Lagrange multiplier. According to the structure Theorem 2.1, we denote by l_1^L, l_2^L the forms defining the first and second derivatives of L at the unit disk Ω with its boundary Γ. We denote $u = u_\Omega$.

Lemma 5.1 *For all $h \in \mathcal{C}^3(\Gamma)$,*

$$l_1^L(h) = \int_\Gamma (g(\frac{1}{4}) - \Lambda)h,$$

$$l_2^L(h,h) = \frac{1}{4}\int_\Omega g''(u)H(h)^2 + \int_\Gamma [g(\frac{1}{4}) - \Lambda + \frac{1}{2}g'(0)]h^2 - 2[g(\frac{1}{4}) - g(0)]hD(h),$$

where $D(h)$ is defined as in the previous sections and $H(h)$ denotes the harmonic extension of h to the unit disk Ω. If moreover Ω is an equilibrium shape (i.e.$l_1^L \equiv 0$) and if h is given by

$$h = \sum_{n\geq 1}(a_n \cos n\theta + b_n \sin n\theta), \tag{5.1}$$

then

$$l_2^L(h,h) = \pi \sum_{n=2}^{+\infty} n(a_n^2 + b_n^2)\Big[\int_0^1 g'(u)r^{2n-1}dr + 2[g(0) - g(\frac{1}{4})]\Big]. \tag{5.2}$$

We postpone the proof of this lemma which requires some serious computations (as often for second shape derivatives). We will give all necessary details even if similar computations may be found here and there in the literature (see e.g. [1],[11],[9],[10]).

According to the expression of l_1^L, *the disk* Ω *is a critical shape of* $L(\cdot)$ when we choose $\Lambda := g(\frac{1}{4})$ and therefore a critical shape of $\omega \to \int_\omega g(u_\omega)$ under the constraint $S(\omega) = S(\Omega)$.

We will denote $Q(h) = l_2^L(h,h)$. A main remark is that Q vanishes on the subspace spanned by $\{cos\theta, sin\theta\}$ (i.e such that $a_1 = b_1 = 0$). This comes from the fact that the minimization problem is invariant by translations (here $f \equiv 1$).

It is clear on the expression (5.2) that Q is continuous for the norm of $H^{1/2}(\Gamma)$. Thus the optimal space to look for the coercivity of Q is here

$$Z_{00}^{1/2} = \{h \in H^{1/2}(\Gamma); \int_\Gamma h = 0,\ \int_\Gamma h\, cos\theta = 0,\ \int_\Gamma h\, sin\theta = 0\}.$$

Note that by the results in [5], [4], this coercivity implies the existence of a local minimum for regular (e.g. C^3) perturbations of Ω preserving the volume and the barycentre.

We will also look at an example where coercivity occurs only for the L^2 norm.

For $a \in (0,1)$, we introduce the sets

$$\Omega_a = \Omega \backslash \{(x,y) \in \mathbb{R}^2; x^2 + y^2 = a^2\}.$$

Obviously, Ω_a has the same Lebesgue measure and the same center as the disk Ω itself. We will look at variations of $a \to \int_{\Omega_a} g(u_{\Omega_a})$ for a close to 1 and for a close to 0: in these cases, the Ω_a's are "close" to Ω but, obviously are not "regular" perturbations of Ω.

Theorem 5.2 *There exists a function* g *and* $\eta > 0$ *such that*

$$\forall h \in Z_{00}^{1/2},\ Q(h) \geq \eta \|h\|^2_{L^2(\Gamma)}$$

and such that, for $a \in (0,1)$ *close enough to* 1

$$L(\Omega_a) < L(\Omega).$$

There exists a function g *and* $\eta > 0$ *such that*

$$\forall h \in Z_{00}^{1/2},\ Q(h) \geq \eta \|h\|^2_{H^{1/2}(\Gamma)}$$

and such that, for $a \in (0,1)$ *close enough to* 0

$$L(\Omega_a) < L(\Omega).$$

Remark 5.1 As announced in the introduction, these examples prove that, although the second shape derivative at the unit disk is strictly positive, there are open subsets, "close" to the unit disk (in some sense) and satisfying the constraints, for which the functional is strictly less than for the disk.

The second example is somehow more satisfactory since the coercivity holds in the natural space $H^{1/2}$. But, in this case, even the first derivative is pathological: note that the functional is not differentiable with respect to the parameter

a at $a = 0$, although the classical first shape derivative does vanish for the disk. This shows that the so-called "topological" derivative should be also taken into account to decide about the existence of a true minimum.

The first example is more regular in a way, since the first derivative with respect to a does exist and vanishes at $a = 1$. Here again, the classical derivative turns out to be insufficient to decide about minima.

Remark 5.2 It is easy see that we may perturb the first example to obtain a similar counterexample with an annulus close to the disk Ω and of the form

$$\Omega_{a,b,c} = \{x \in \mathbb{R}^2;\ |x| \in [0, a) \cup (b, c)\},$$

where a, b, c are close to 1, $a < b < c$ and $c^2 - b^2 + a^2 = 1$. Note that $\Omega_{a,b,c}$ are then even C^∞ shapes!

Proof: Let $C_a = (a^2 - 1)/4 \ln a$. Then $u_a = u_{\Omega_a}$ is given by

$$\forall a \in]0,1[: \quad u_a(r) = \begin{cases} \dfrac{a^2 - r^2}{4} & \text{if } 0 \le r < a, \\ \dfrac{1 - r^2}{4} + C_a \ln r & \text{if } a < r \le 1. \end{cases}$$

and $u_1(r) = (1 - r^2)/4, \forall r \in [0, 1]$.
First example: Let $c \in \mathbb{R}$ and $g(x) = -x^2 + cx$. By (5.2),

$$Q(h) = -\frac{\pi}{2} \sum_{n \ge 2} \big[\frac{1}{2n+2} - c + n(c - \frac{1}{4})\big](a_n^2 + b_n^2). \tag{5.3}$$

An explicit computation of $J(u_1)$ and $J(u_a)$ gives

$$\frac{1}{2\pi}[J(u_1) - J(u_a)] = \frac{(1 - a^2)C_a}{4}(c - \frac{3(1 + a^2)}{8} + C_a), \tag{5.4}$$

so that, as a tends to 1, if $c \neq 1/4$,

$$J(u_1) - J(u_a) \quad \sim \quad \frac{\pi}{2}(1 - a)(c - \frac{1}{4}).$$

Let us precisely choose $c = 1/4$. Then,

$$Q(h) = \frac{\pi}{2} \sum_{n \ge 2} (\frac{1}{4} - \frac{1}{2n+2})(a_n^2 + b_n^2) \ge \frac{\pi}{24} \sum_{n \ge 2} (a_n^2 + b_n^2).$$

On the other hand, from (5.4), we see that, as a tend to 1

$$J(u_1) - J(u_a) \quad \sim \quad \pi(1 - a)^2/8,$$

so that $J(u_1) > J(u_a)$ when a is close to 1.

Second example: Note first that, for all choice of g, we have

$$\frac{d}{da} J(u_a) = \pi \int_0^a g'(u_a) a\, r\, dr + 2\pi C_a' \int_a^1 g'(u_a) r \ln r\, dr,$$

so that, as a tends to 0

$$\frac{d}{da}J(u_a) \quad \sim \quad \frac{\pi}{2a(\ln a)^2}\int_0^1 g'(u_0) r\ln r\,dr. \tag{5.5}$$

The second example consists in choosing g in such a way that $g(0) = 0$ and

$$\int_0^1 g'(u_0)r\ln r dr < 0,\ \int_0^1 g'(u_0)r^{2n-1}dr - 2g(\frac{1}{4}) \geq \eta > 0,\ \forall n \geq 2. \tag{5.6}$$

For this, we choose $\alpha > 0, b \in (0, 1/4)$ such that

$$\alpha\, b > 1,\ \frac{1}{4b} > \frac{\alpha}{4} - 1 > \frac{-1}{\ln(1-4b)}, \tag{5.7}$$

(this is for instance realized by $\alpha = 10,\ b = 1/8$), and we set $g(0) = 0$ and

$$g'(s) = \begin{cases} -\alpha & \text{if} \quad 0 \leq s \leq b, \\ 4(\alpha b - 1)/(1-4b) & \text{if} \quad b \leq s \leq \dfrac{1}{4}. \end{cases}$$

We check that the sequence

$$n \geq 1 \to \alpha_n = \frac{1}{2n}[(\alpha-4)(1-4b)^{(n-1)} - \alpha],$$

is stricly increasing and that

$$\forall n \geq 2,\ \int_0^1 g'(u_0)r^{2n-1}dr = \alpha_n,\ \ 2\,g(1/4) = \alpha_1 = -2,$$

$$\int_0^1 g'(u_0)r\ln r dr = (\frac{\alpha}{4} - 1)\ln(1-4b) + 1.$$

We verify that assumption (5.7) implies (5.6).

Since g in only Lipschitz continuous, we may approximate g by C^∞-functions g_p converging uniformly on $(0, 1/4)$ to g as $p \to +\infty$ and such that g_p' is uniformly bounded and converges a.e. to g'. Then, inequalities (5.6) are satisfied by g_p for p large enough.

Proof of Lemma 5.1: We use the same normal deformations as in (2.2) which are of the form

$$T_t x = x + t\xi(x),\ \xi^\tau = 0,\ \xi\cdot\nu = h,\ \Omega_t = T_t(\Omega).$$

Then, if $l(t) = L(\Omega_t)$, we have (see (2.3))

$$l_1^L(h) = l'(0),\ l_2^L(h,h) = l''(0).$$

Since, we already know the derivatives of $S(\cdot)$, we only have to compute $e'(0), e''(0)$ where $e(t) = \int_{\Omega_t} g(u_t)$ and u_t is solution of

$$-\Delta u_t = 1 \text{ in } \Omega_t, \quad u_t = 0 \text{ on } \partial\Omega_t. \tag{5.8}$$

We have $u_0(r) = u(r) = (1-r^2)/4$, $\nabla u = -\nu/2$ *on* Γ. Note that $\partial_{t_{|t=0}} u_t$ is solution of

$$\Delta(\partial_{t_{|t=0}} u_t) = 0 \quad \text{in} \quad \Omega, \quad \partial_{t_{|t=0}} u_t = \frac{1}{2} h \quad \text{on} \quad \Gamma. \tag{5.9}$$

Thus, with the notation of Lemma 5.1, $\partial_{t_{|t=0}} u_t = \frac{1}{2} H(h)$. Differentiating twice at the boundary leads also to

$$\partial^2_{tt_{|t=0}} u_t = -2h \frac{\partial(\partial_{t_{|t=0}} u_t)}{\partial \nu} + \frac{1}{2} h^2 = -hD(h) + h^2/2 \text{ on } \Gamma. \tag{5.10}$$

We will repeatedly use the following classical derivation formula which is valid for regular $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$

$$\frac{d}{dt} \int_{\Omega_t} f(t) = \int_{\Omega_t} [\frac{\partial f}{\partial t}(t) + \nabla \cdot (f(t) \xi \circ \Pi(t))] \tag{5.11}$$

where $\Pi(t) := (I + t\xi)^{-1}$. We apply it to $e(t)$.

$$e'(t) = \int_{\Omega_t} g'(u_t) \partial_t u_t + \int_{\Omega_t} \nabla \cdot (g(u_t) \xi \circ \Pi(t)). \tag{5.12}$$

In particular,

$$e'(0) = \int_{\Omega} g'(u) \partial_{t_{|t=0}} u_t + g(0) \int_{\Gamma_0} h. \tag{5.13}$$

Let us introduce the solution p of the adjoint problem

$$-\Delta p = g'(u) \text{ in } \Omega, \qquad p = 0 \text{ on } \Gamma,$$

which satisfies

$$p'(1) = -\int_0^1 r g'((1-r^2)/4)\, dr = 2[g(0) - g(1/4)]. \tag{5.14}$$

We have

$$\int_{\Omega} g'(u) \partial_{t_{|t=0}} u_t = -\int_{\Omega} \Delta p (\partial_{t_{|t=0}} u_t) = -\int_{\Gamma} (\partial_{t_{|t=0}} u_t) p'(1). \tag{5.15}$$

Now, the first part of Lemma 5.1 follows from (5.13,5.15,5.14,5.9,2.9).

Differentiating (5.12) at $t = 0$ leads to

$$e''(0) = \int_{\Omega} g''(u) (\partial_{t_{|t=0}} u_t)^2 + \int_{\Omega} g'(u) \partial^2_{tt_{|t=0}} u(t) + \int_{\Gamma} h g'(u) \partial_{t_{|t=0}} u_t + \frac{d}{dt}_{|t=0} \int_{\Omega_t} \nabla \cdot (g(u_t) \xi \circ \Pi(t)). \tag{5.16}$$

For the last integral I_4 of (5.16), we obtain by using (5.11), integration by parts and $\partial_{t_{|_{t=0}}} (\xi \circ \Pi(t)) = -D\xi\, \xi$:

$$I_4 = \int_{\Gamma} \nu \cdot [g'(0) \xi \partial_{t_{|_{t=0}}} u_t - g(0) D\xi\, \xi] + h \nabla \cdot (g(u)\xi).$$

But, recall that $\xi = \tilde{h}\tilde{\nu}$ where the norm of $\tilde{\nu}$ is equal to 1, so that, on Γ

$$\nu \cdot D\tilde{\nu} = 0,\ \nu \cdot D\xi \cdot \nu - \nabla \cdot \xi = -h\nabla \cdot \tilde{\nu} = -h.$$

Adding everything yields

$$I_4 = \frac{d}{dt}_{|t=0} \int_{\Omega_t} \nabla \cdot (g(u_t)\xi \circ \Pi(t)) = g(0) \int_\Gamma h^2. \tag{5.17}$$

For the third integral of (5.16), we have

$$\int_\Gamma hg'(u)\partial_{t_{|t=0}} u_t = \frac{1}{2} g'(0) \int_{\Gamma_0} h^2. \tag{5.18}$$

For the second integral of (5.16), by double integration by parts and using (5.10,5.14), we obtain

$$\begin{aligned} \int_\Omega g'(u)\partial^2_{tt_{|t=0}} u_t = -\int_\Omega \Delta p \partial^2_{tt_{|t=0}} u_t = -\int_\Gamma p'(1)\partial^2_{tt_{|t=0}} \\ = [g(\frac{1}{4}) - g(0)] \int_\Gamma h^2 - 2hD(h). \end{aligned} \tag{5.19}$$

Using (5.16), the expressions of the three integrals above and (2.9), we obtain the second statement of Lemma 5.1.

If now h is given by its Fourier series expansion (5.1), we have that

$$H(h) = \sum_{n \geq 1} r^n (a_n \cos n\theta + b_n \sin n\theta) \quad \text{on} \quad \Omega. \tag{5.20}$$

Elementary computations lead to

$$\begin{aligned} &\int_\Gamma h^2 = \pi \sum_{n\geq 1} (a_n^2 + b_n^2), \quad \int_\Gamma hD(h) = \pi \sum_{n \geq 1} n(a_n^2 + b_n^2), \\ &\int_\Omega g''(u)H(h)^2 = 2\pi \sum_{n \geq 1} (a_n^2 + b_n^2)[-g'(0) + 2n \int_0^1 g'(u_0) r^{2n-1} dr]. \end{aligned}$$

We deduce (5.2) (note that the term associed to $n = 1$ vanishes).

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